An optimal control formulation for shape-matching in augmented surgery

Guillaume Mestdagh, Yannick Privat and Stéphane Cotin

Institut de recherche mathématique avancée
UMR 7501 Université de Strasbourg et CNRS
7 rue René Descartes
67000 Strasbourg, France

Tel: 03 68 85 02 78 - **Email:** guillaume.mestdagh@unistra.fr

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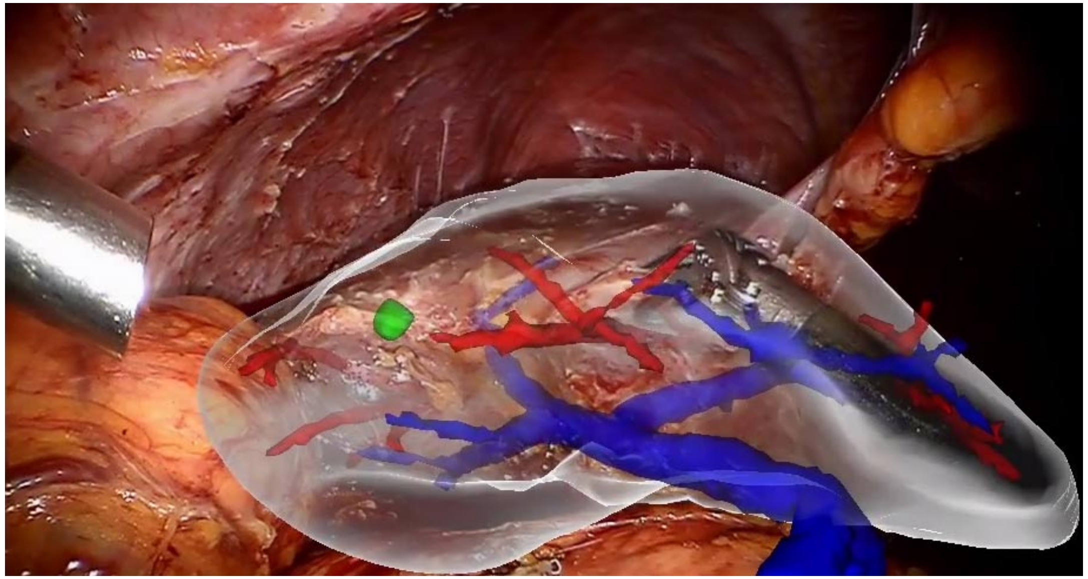
Introduction



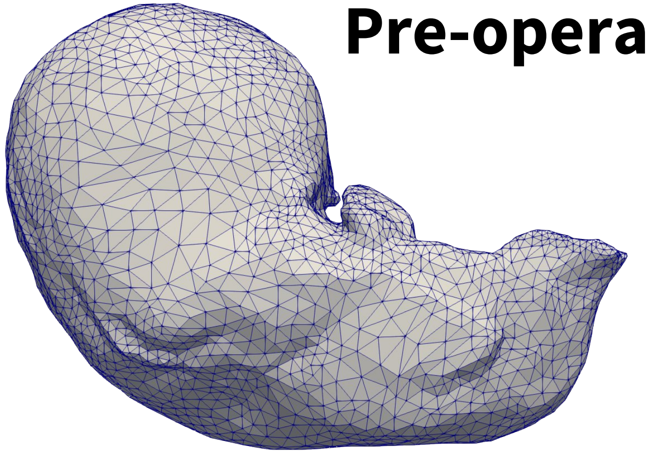
Augmented reality image during liver surgery

Inria, 2018

- Reconstruct organ displacement from intra-operative data acquisition
- Superpose a 3D view onto organ image
- Track tumor location in real-time

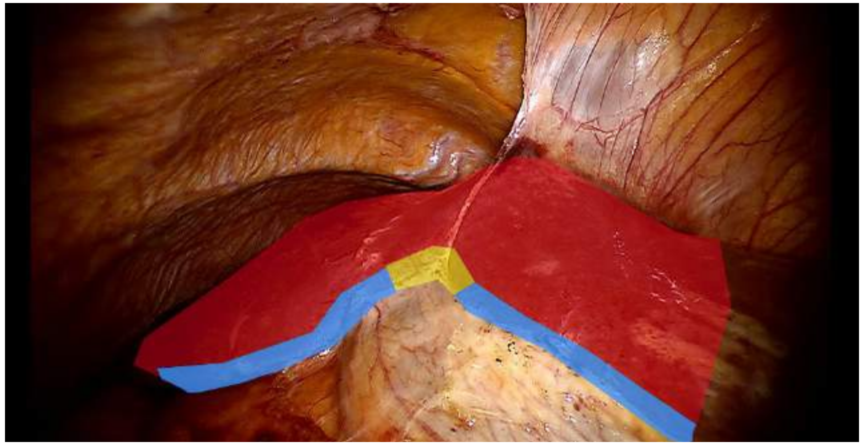
Shape-matching problem

Pre-operative data



3D model of the liver in its initial configuration

Intra-operative data

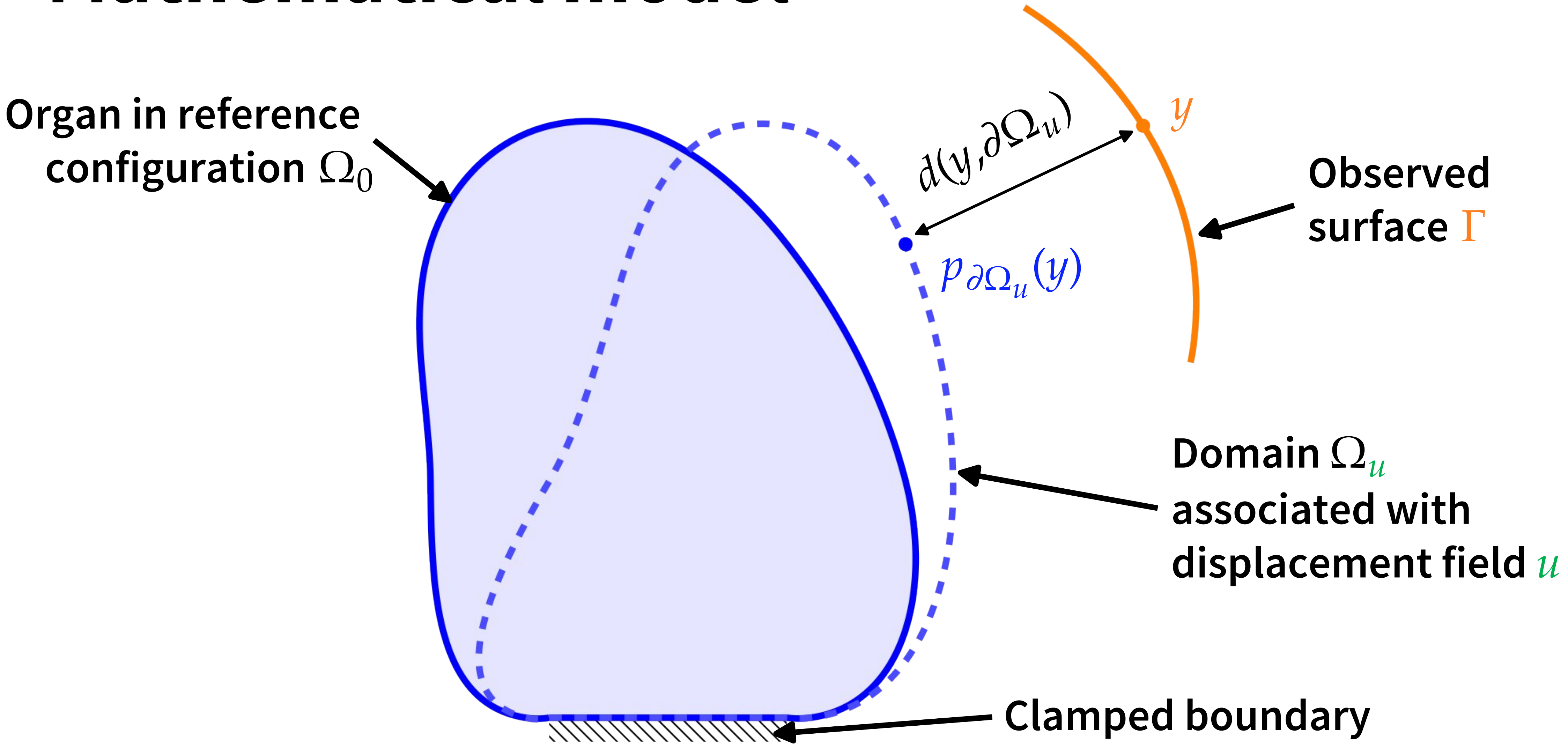


Partial location of the organ surface (R. Plantefève, 2016)

Objective: deform the mesh to match the observed surface

Plantefève et al., *Patient-specific Biomechanical Modeling for Guidance during Minimally-invasive Hepatic Surgery*, 2016

Mathematical model



The liver: an elastic solid

Notation

Displacement field $u \in H^1_{\mathrm{D}}(\Omega_0)$

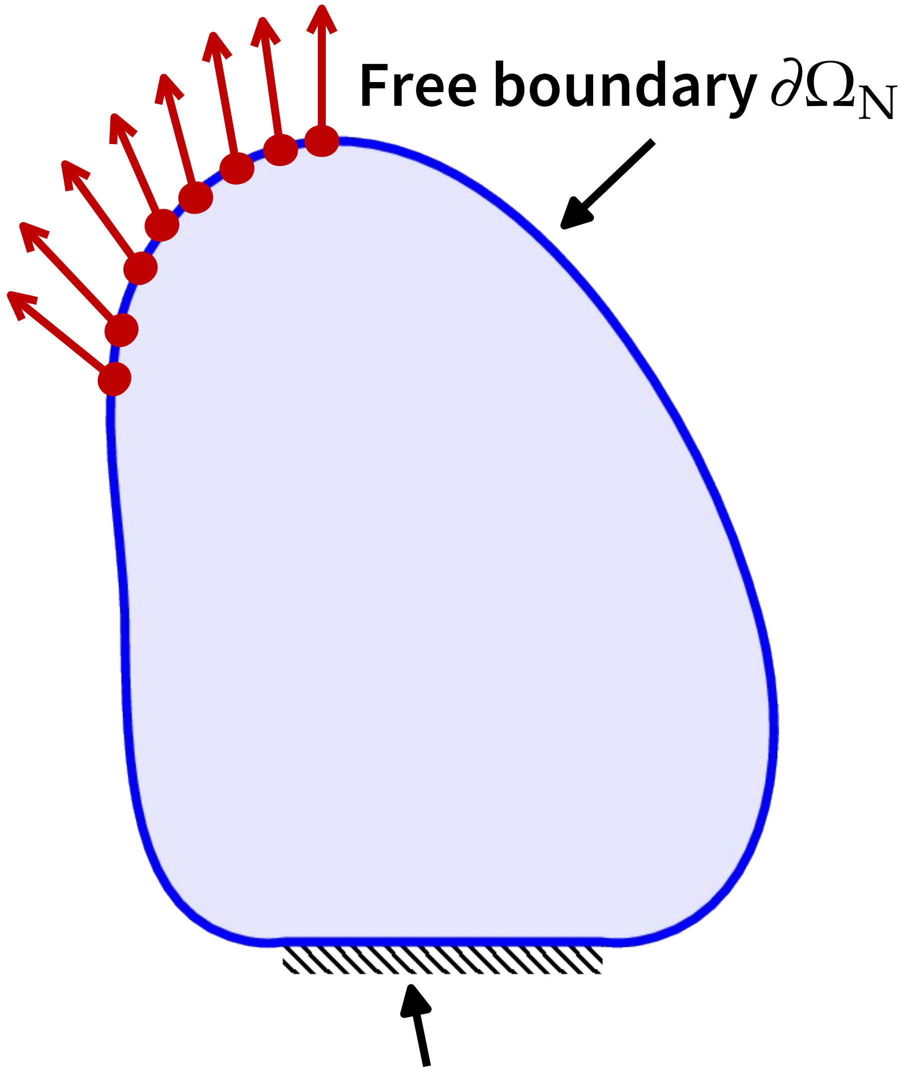
Surface loading $g \in L^2(\partial\Omega_{\mathrm{N}})$

Elasticity equation

Displacement field

$$\begin{cases} \operatorname{div}(\sigma(u)) = 0 & \text{in } \Omega_0 \\ u = 0 & \text{on } \partial\Omega_{\mathrm{D}} \\ \sigma(u)\cdot n = g & \text{on } \partial\Omega_{\mathrm{N}} \end{cases}$$

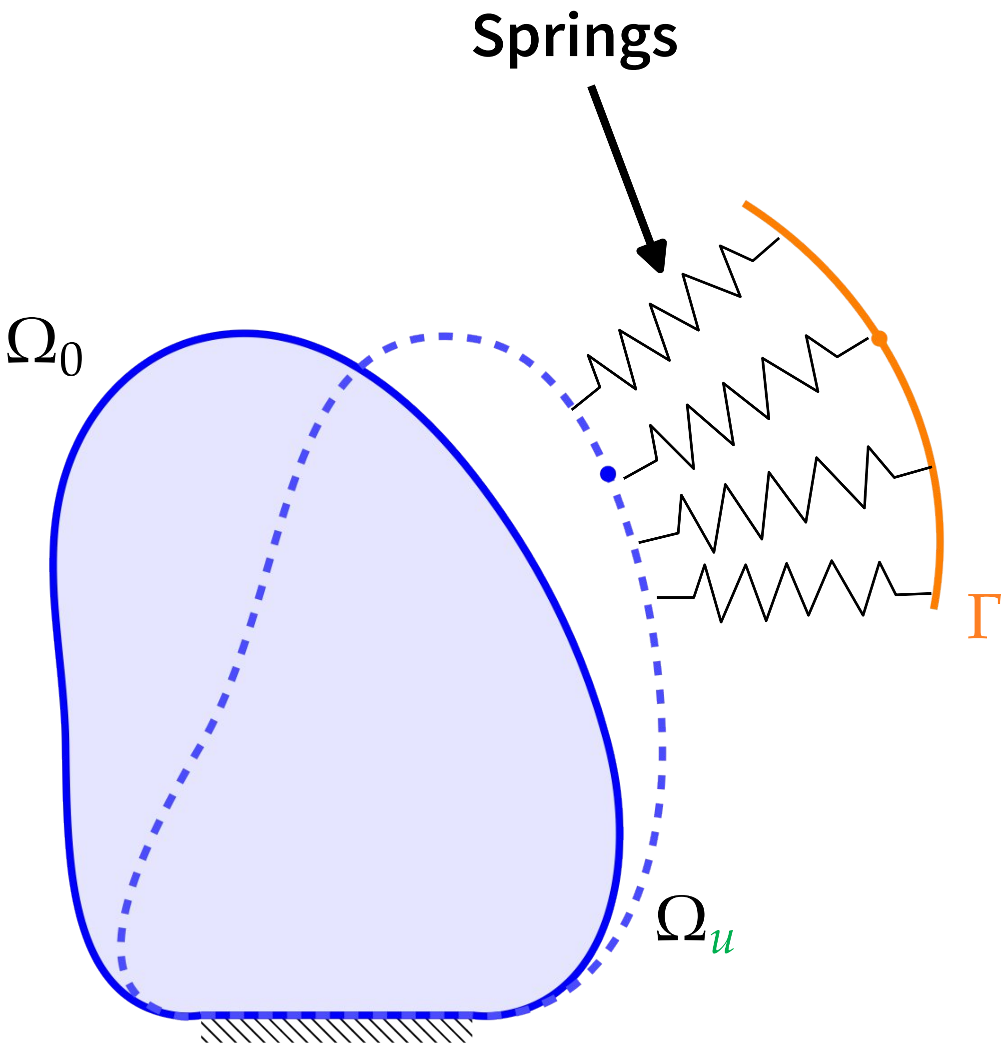
Surface loading



State of the art

Use artificial forces

- Add springs between Γ and organ boundary
- Solve static elasticity problem to compute displacement
- Progressively increase spring stiffness



Haouchine et al., *Impact of Soft Tissue Heterogeneity on Augmented Reality for Liver Surgery*, 2015

Optimal control problem

Find a surface loading field g solution of

Discrepancy with data

Pointwise constraint on surface loading

$$\min J\left(u_g\right) + R(g) \quad \text{s.c} \quad \|g\| \leq M \textbf{ sur } \partial\Omega_{\mathrm{N}}$$

Penalization term

u_g : elastic displacement created by g.

Why an optimal control problem

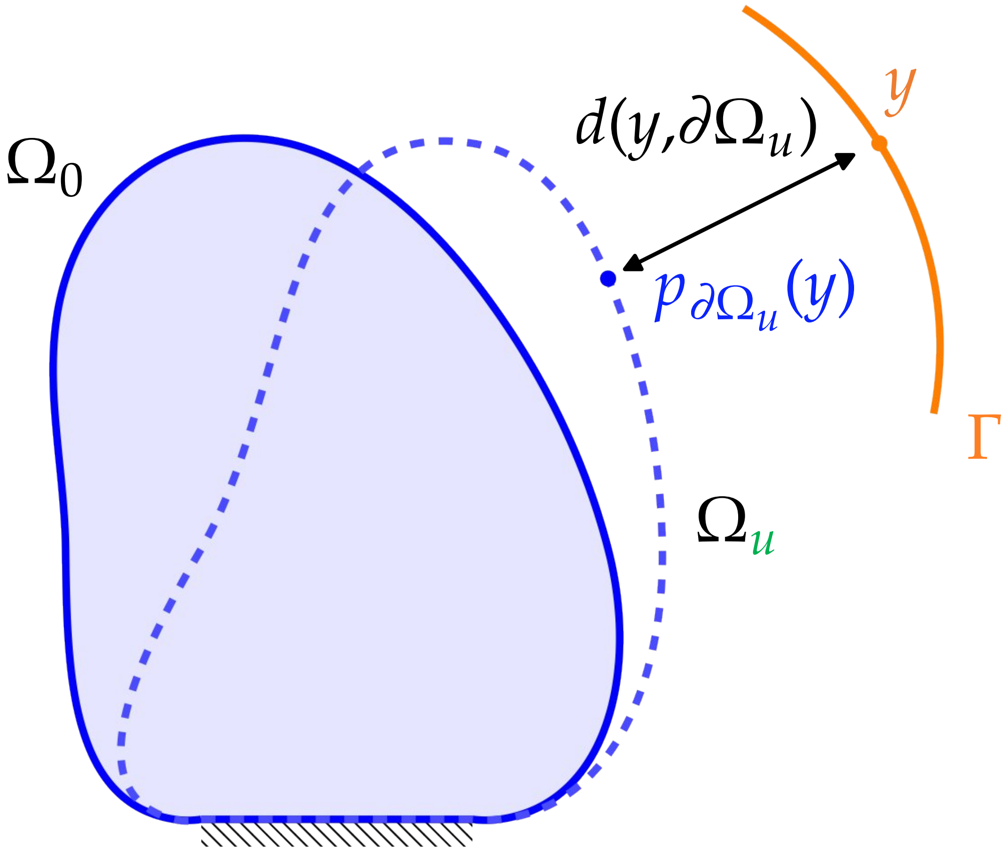
- Reconstruct realistic surface force field instead of creating artificial forces
- Add physical or statistical information with penalties or constraints
- Use generic optimization tools to study and solve numerically the problem

A functional to measure registration quality

(Nearly-) shape functional

$$J(u) = \frac{1}{2}\int_{\Gamma} d^2(y, \partial\Omega_u)\ \mathrm{d}y$$

- $J(u) = 0$ **only when registration is successful (i.e** $\Gamma \in \partial\Omega_u$**)**
- **Flexible : can be adapted with respect to data uncertainty**



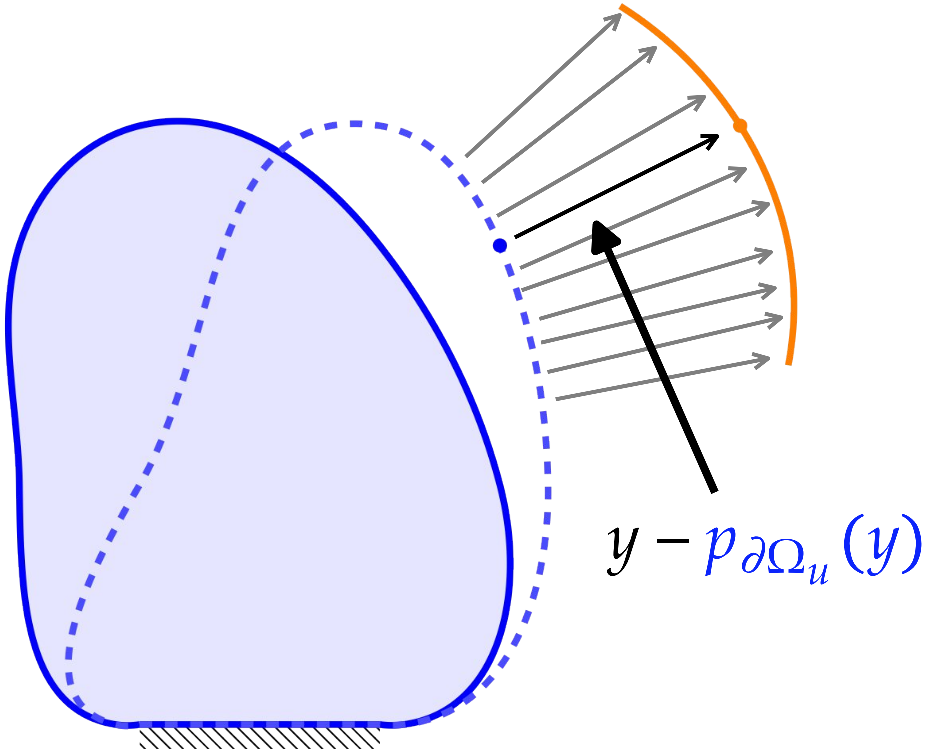
Functional : differentiability

Proposition

J has directional derivatives in $L^2(\partial\Omega_N)$

Compute descent directions

- Use linear elasticity inner product = transform L^2 gradient into forces
- Very similar to spring approach



L^2 gradient of J

Danskin, *The theory of Min-Max and its application to weapon allocation problems*, 1967

2 An optimal control formulation

Theoretical results

Existence of solutions

- **Toy problem with simpler model :** $\min J\left(u_g\right)$ s.c $\begin{cases} \Delta u + u = 0 & \text{dans } \Omega_0 \\ \partial_n u = g & \text{sur } \partial\Omega \end{cases}$

- **Proposition :** Problem has at least one solution

Optimality conditions

- Useful to compute descent directions

- Involve adjoint state (see later)

Lieberman, *The conormal derivative problem for elliptic equations of the variational type, 1983*

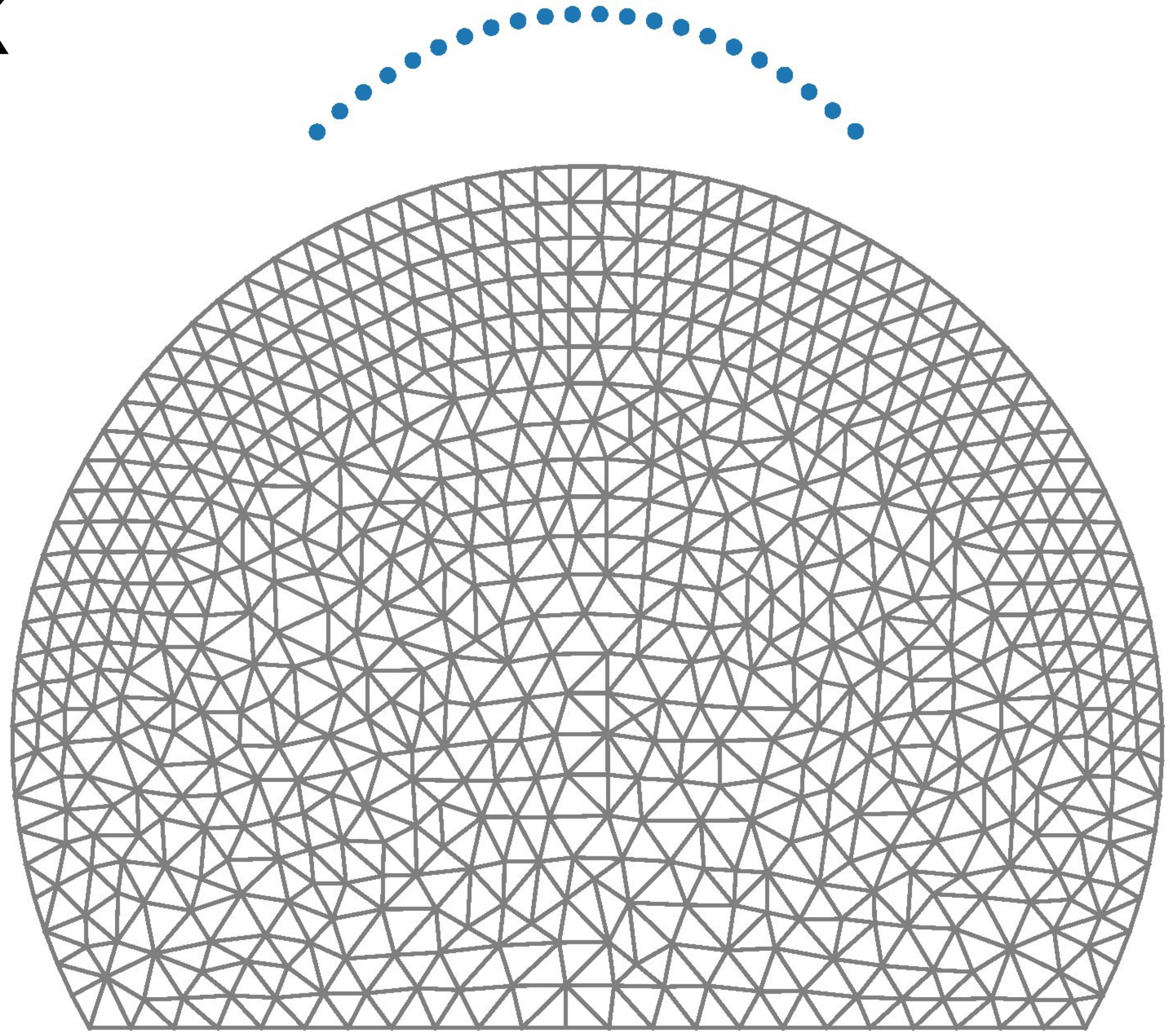
Numerical framework

- The organ : a mesh
- The target : a point cloud
- Vector fields : P1 finite elements functions
- Linear elasticity equation

$$\mathbf{A}\mathbf{u} = \mathbf{S}\mathbf{g}$$

Stiffness matrix

Boundary measure matrix

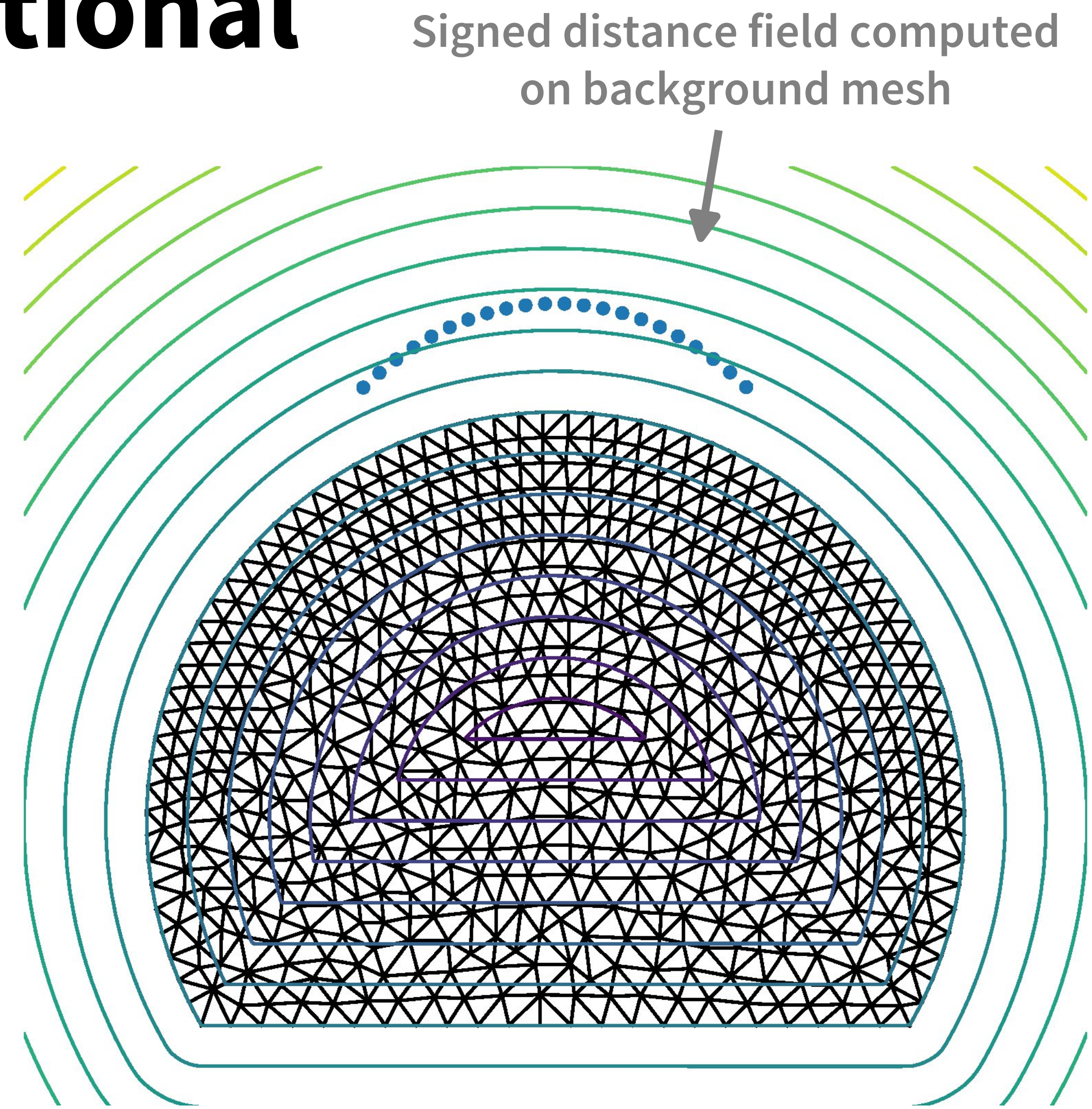


Compute discrete functional

$$J(\mathbf{u}) = \frac{1}{2}\sum_{y\in\Gamma} d^2(y, \partial\Omega_{\mathbf{u}})$$

Difficulty
Many orthogonal projections onto mesh boundary

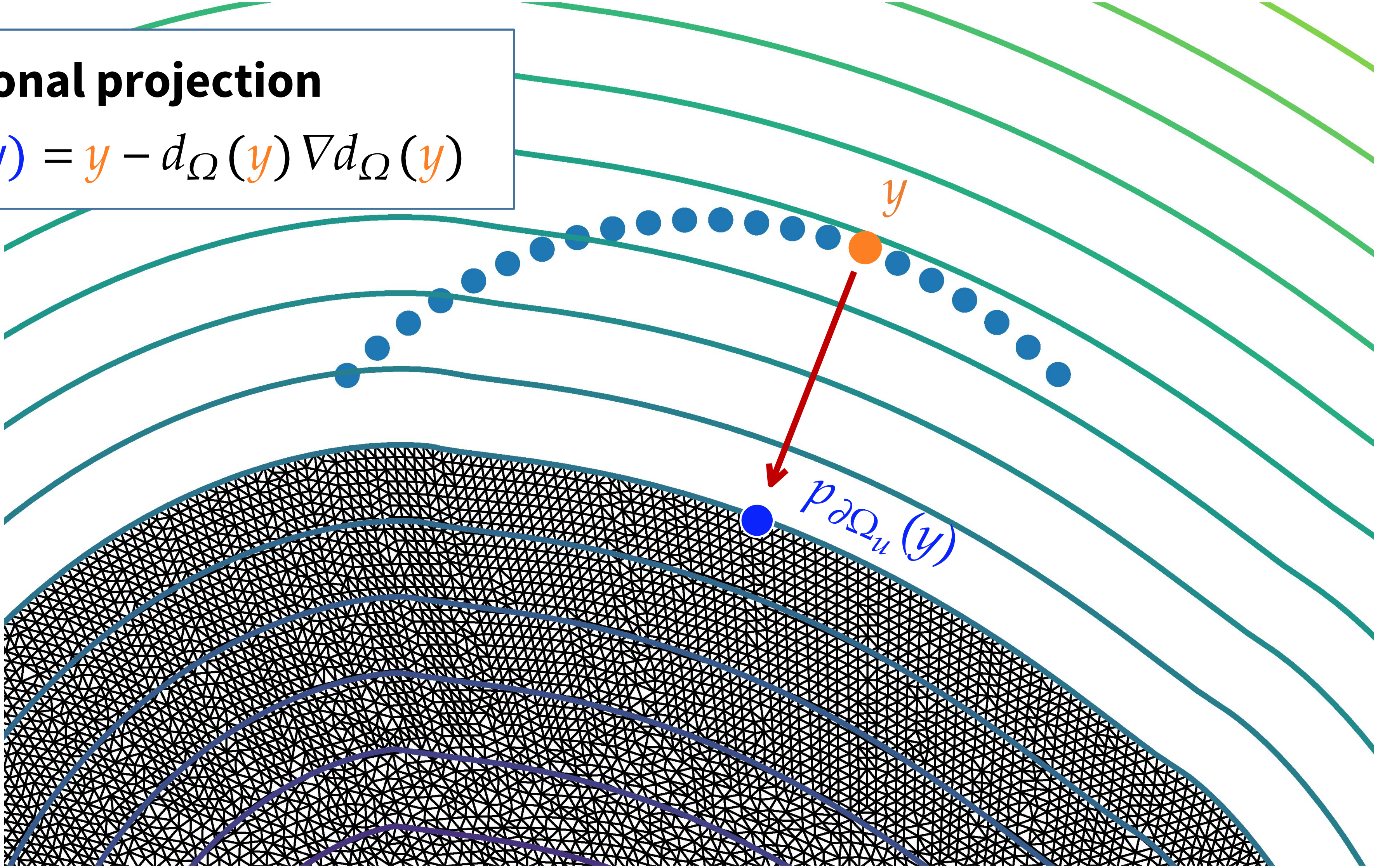
Considered solution
Compute a signed distance field



Compute discrete functional

Orthogonal projection

$$p_{\partial\Omega_u}(y) = y - d_\Omega(y)\nabla d_\Omega(y)$$



3 Numerical aspects

Minimization : adjoint method

Compute objective gradient

$$F(\mathbf{g}) = J\left(\mathbf{u_g}\right) + R(\mathbf{g})$$

1. Solve direct problem $\quad \mathbf{A}\mathbf{u} = \mathbf{S}\mathbf{g}$
2. Solve adjoint problem $\quad \mathbf{A}\mathbf{p} = \nabla J(\mathbf{u})$
3. Compute gradient $\quad \nabla F(\mathbf{g}) = \mathbf{S}^{\mathrm{T}}\mathbf{p} + \nabla R(\mathbf{g})$

Matrix formulation

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{g}}\left[J\left(\mathbf{u_g}\right)\right] = \frac{\mathrm{d}}{\mathrm{d}\mathbf{g}}\left[J\left(\mathbf{A}^{-1}\,\mathbf{S}\mathbf{g}\right)\right] = \mathbf{S}^{\mathrm{T}}\overbrace{\mathbf{A}^{-\mathrm{T}}\,\nabla J\left(\mathbf{A}^{-1}\,\mathbf{S}\mathbf{g}\right)}^{\mathbf{p}}$$

Minimization : gradient descent

Iteration

1. Current iterate : $\mathbf{g}_k$

2. Compute gradient $\nabla F(\mathbf{g}_k)$ using adjoint method

3. Choose stepsize α_k which makes objective function decrease

4. Compute next iterate $\mathbf{g}_{k+1} = \mathbf{g}_k - \alpha_k \nabla F(\mathbf{g})$

Handling a noisy point cloud

Difficulty

Error on intra-operative data

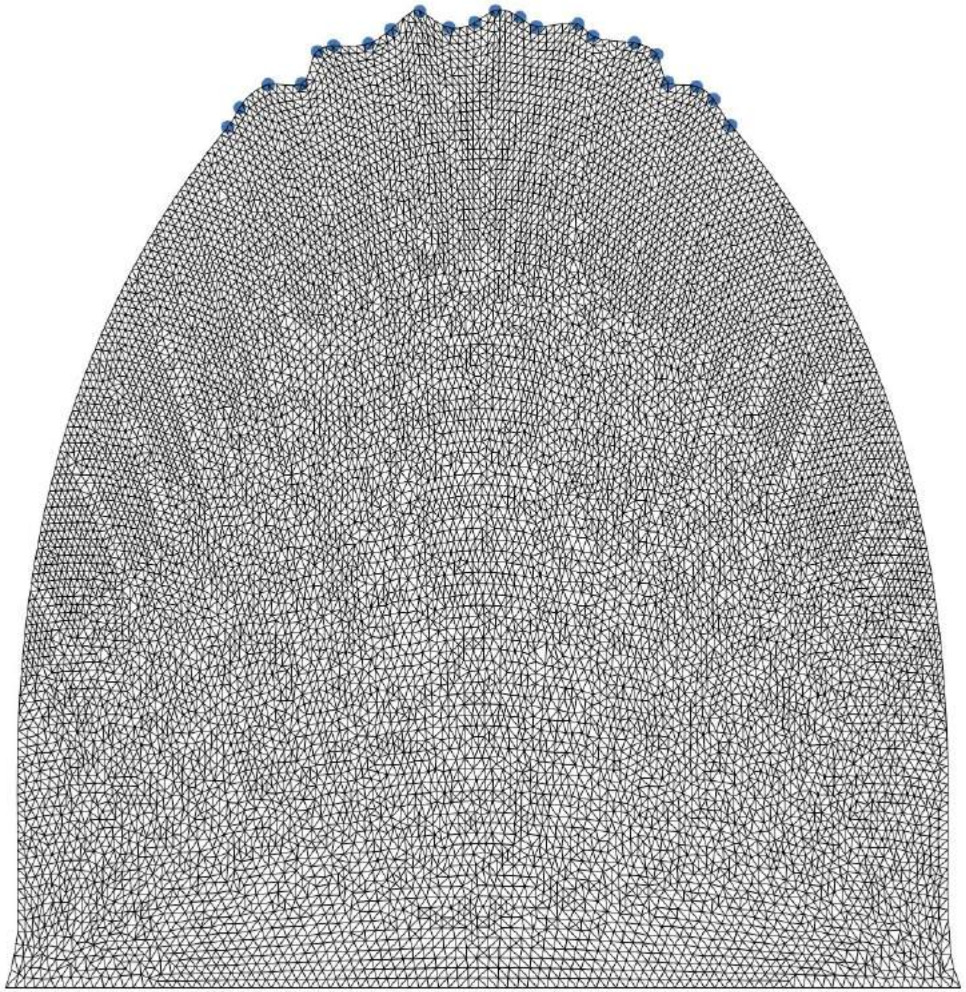
Considered solutions

- Regularized problem

$$\min J(\mathbf{u_g}) + \frac{1}{2}\|\mathbf{g}\|_{L^2}^2$$

- Problem with poinwise constraints

$$\min J(\mathbf{u_g}) \text{ s.c } \qquad \|\mathbf{g}\|_{L^\infty} \leq M$$

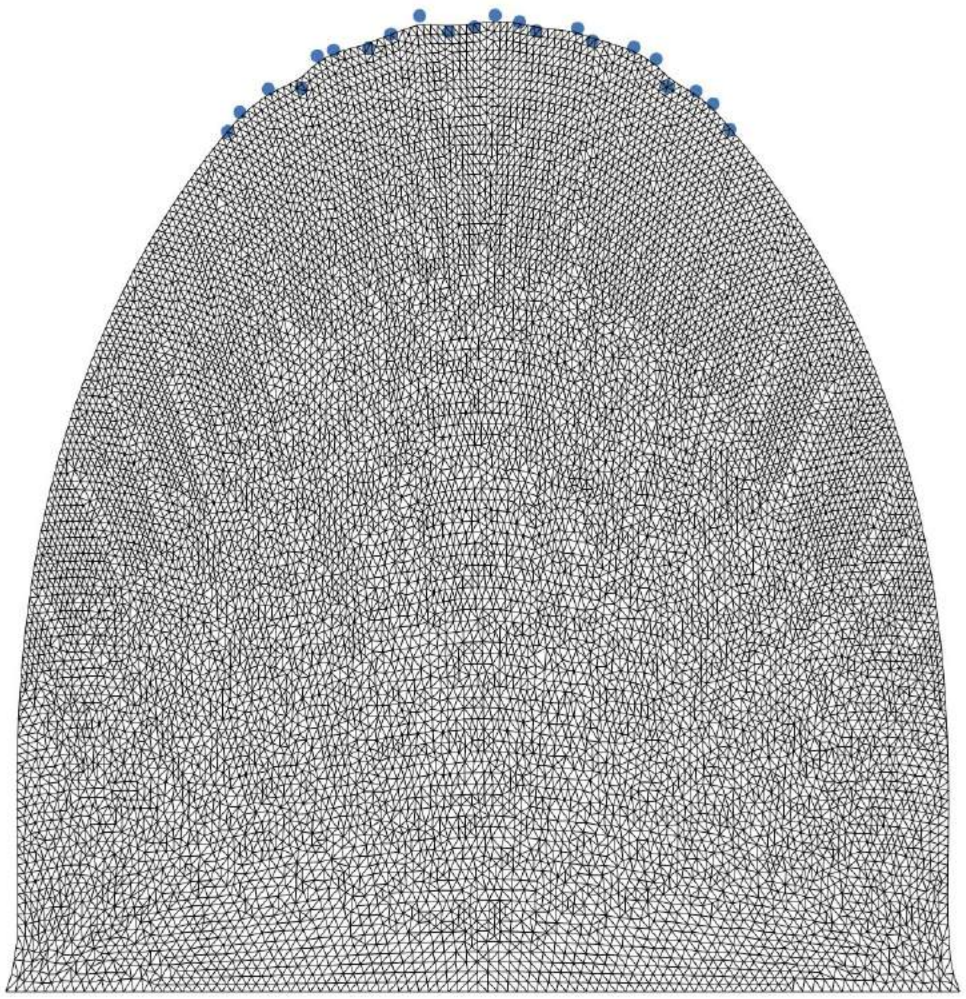


Regularized problem

Regularized problem

$$\min \; J\left(\mathbf{u}_{\mathbf{g}}\right) + \frac{1}{2}\|\mathbf{g}\|_{L^2}^2$$

- Penalize control global norm
- Unconstrained problem
- Problem is more convex and more coercive

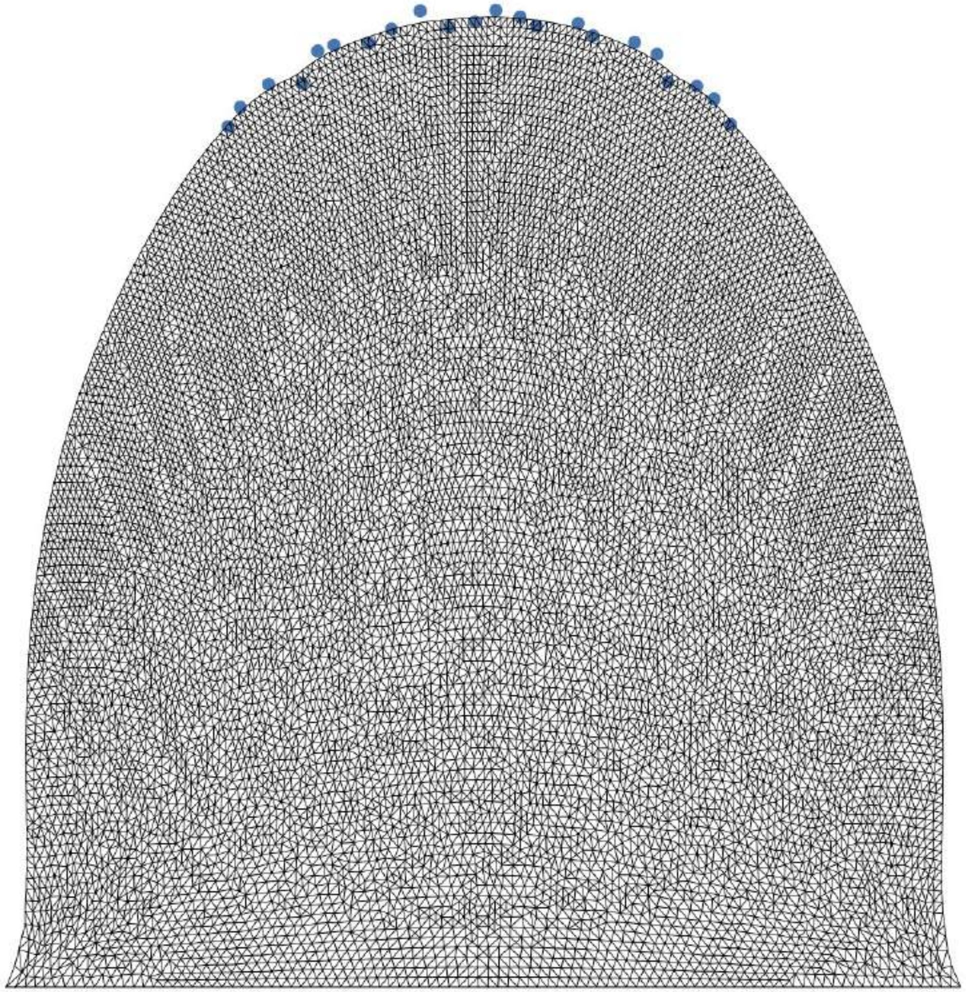


Problem with pointwise constraint

Constrained problem

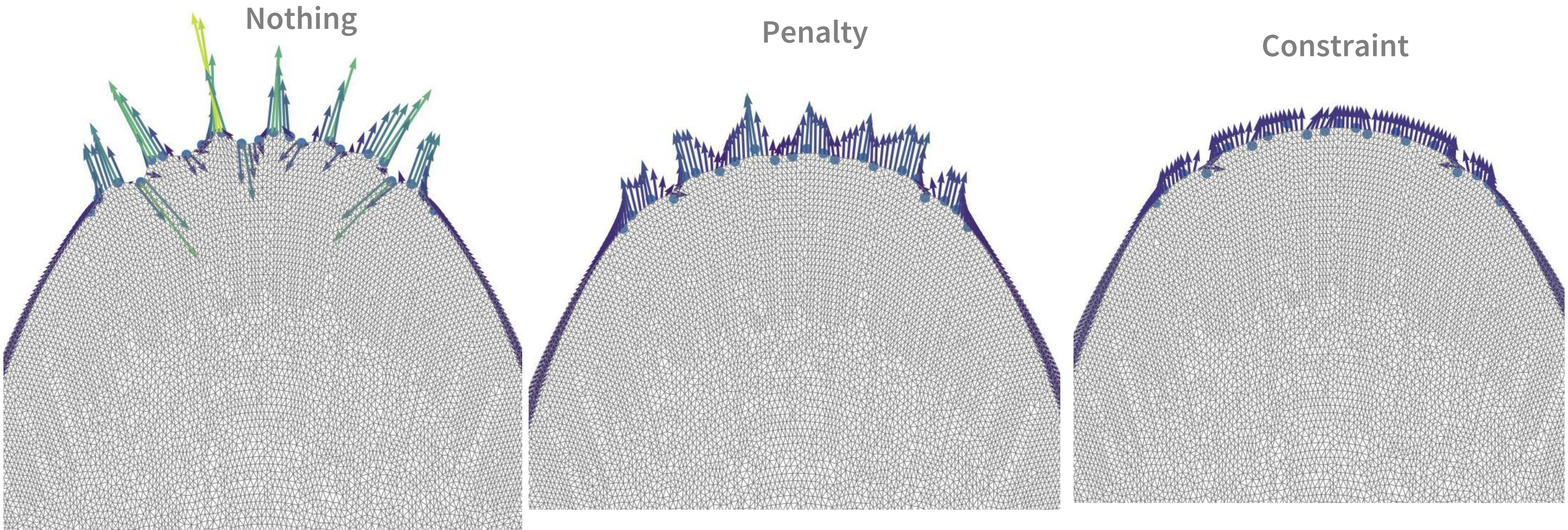
$$\min J\left(\mathbf{u_g}\right) \text{ s.c } \qquad \|\mathbf{g}\|_{L^\infty} \leq M$$

- Pointwise constraint on control
- Physical meaning
- Coherent with existence theory



3 Numerical aspects

Control regularity



4 Conclusion and ongoing work

Optimal control in SOFA

- Create SOFA plugin to handle and solve optimal control problems
- Implement classes for optimization problems and algorithms
- Interact with other projects (neural networks, functional maps)
- Solve specific problems in liver registration

Conclusion

Advantages of optimal control formulation

- Generic tools to study and solve problem

- Easily add physical information into problem

What's next

- Computation on a real-life problem

- Implement efficient program