Séminaire MLMS

ϕ-FEM :

A fictitious domain method for finite element methods on domains defined by level-sets

M. Duprez[1], V. Lleras[2] and A. Lozinski[3],

[1]Inria, Strasbourg, France
[2]IMAG, Montpellier, France
[3]Laboratoire de Mathématiques de Besançon, France

Outline

- The "standard" Finite Element Method
- Geometrical constraint on the mesh
- Previous Fictitious Domain Methods
- ϕ-FEM
- Conclusions and perspectives

Weak formulation

Strong formulation of the Poisson equation

$$\begin{cases} \text{Find } u \in H^2(\Omega) \text{ s.t. :} \\ -\Delta u = f, \text{ in } \Omega \\ u = 0, \text{ on } \partial\Omega \end{cases}$$

$H^2(\Omega) \approx$ differentiable two times
Multiplying by a "test" function v & by integration by part, one has

Weak formulation

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ s.t. :} \\ \Leftrightarrow \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv \quad \forall v \in H_0^1(\Omega). \end{cases}$$

$H_0^1(\Omega) \approx$ differentiable one times, equal to zero on the boundary

Standard FEM formulation

Weak formulation

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ s.t. :} \\ \Leftrightarrow \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv \quad \forall v \in H_0^1(\Omega). \end{cases}$$

Standard FEM formulation

$$\begin{cases} \text{Find } u_h \in V_h \text{ s.t. :} \\ \Leftrightarrow \int_\Omega \nabla u_h \cdot \nabla v_h = \int_\Omega fv_h \quad \forall v \in V_h. \end{cases}$$

where V_h is a subspace of $H_0^1(\Omega)$ of **finite dimensional**.

Standard FEM formulation

Finite element space

Let $V_h = \langle \psi_k \in H^1_0(\Omega) : k \in \{1, ..., N\}\rangle \subset H^1_0(\Omega)$.

Equivalence with a matrix system

$$\left\{ \begin{array}{l} \text{Find } u_h = \sum U_{hk}\psi_k \in V_h \text{ s.t. :} \\ \int_\Omega \nabla u_h \cdot \nabla \psi_k = \int_\Omega f\psi_k \quad \forall k \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \text{Find } U_h \in \mathbb{R}^N \text{ s.t. :} \\ A_h U_h = F_h \end{array} \right.$$

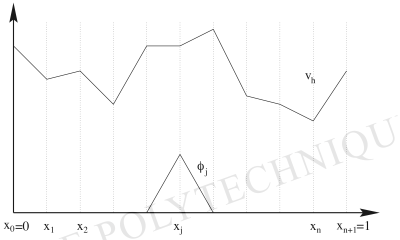
where

$$\left\{ \begin{array}{l} A_h = (\int_\Omega \nabla\psi_k \cdot \nabla\psi_j)_{kj} \\ F_h = (\int_\Omega f\psi_k)_k \\ U_h = (U_{hk})_k \end{array} \right.$$

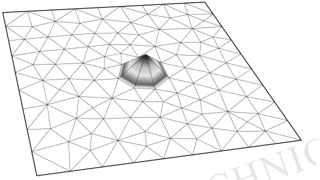
Final solution

$$u_h = \sum U_{hk}\psi_k.$$

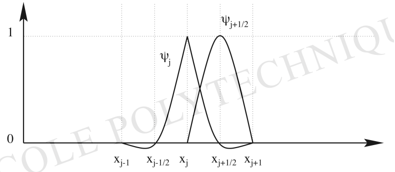
FE space : $V_h = \{$cont. piecewise pol. functions on a **regular mesh**$\}$



Piecewise linear Lagrange FE order 1 ($\mathbb{P}_1$)



Order 1 ($\mathbb{P}_1$), dimension 2



Order 2 ($\mathbb{P}_2$), dimension 1

Why use polynomials ?

because the computation of the coefficient $\int_\Omega \nabla\psi_k \cdot \nabla\psi_l$ of the FE matrix is explicit and **exact**

Important : if ψ_k is not polynomial, this computation is not exact anymore

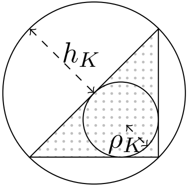
Why a mesh ?

because we can **not good approximate a function** by a polynomial on the whole domain

Regular mesh : Geometrical condition

- **Simplex (triangle, tetrahedron)**
  The standard FEM works under the **Ciarlet condition** [Ciarlet 78']

$$\frac{h_K}{\rho_K} < \gamma$$



- **In general (hexahedron,...)**
  The standard FEM works if the Jacobian matrix to an reference element is positive.

Important remark : If the mesh contains degenerated cells :

- It is not guaranty that standard FEM converges
- The **conditioning number** of the FE matrix is bad

What is the conditioning number ?

Conditioning number of a matrix A :

$$\mathrm{Cond}(\boldsymbol{A}) := \|\boldsymbol{A}\|_2 \|\boldsymbol{A}^{-1}\|_2$$

For symmetric definite positive matrices :

$$\mathrm{Cond}(\boldsymbol{A}) := \frac{\text{bigest eigen value}}{\text{smallest eigen value}} > 1$$

Example

if $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}$ then $\mathrm{Cond}(\boldsymbol{A}) = 1/\varepsilon$.

If ε goes to zero, $\boldsymbol{A}$ goes to a non-invertible matrix

Conclusion

the conditioning number measures the **invertibility of the matrix**.

Poor conditioning of the system matrix

Proposition

- ***Non-degenerated mesh :*** *Suppose that the mesh satisfies the geometrical conditions, then*

$$Cond(\boldsymbol{A}) \leqslant C/h^2$$

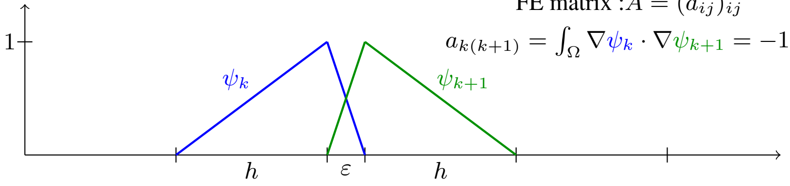
  where h is the size of the cells.

- ***Only one degenerated cell :***

$$Cond(\boldsymbol{A}) \geqslant C/h\varepsilon$$

  where ε is the size the degenerated cell.

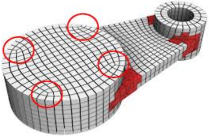
Idea :



FE matrix : $A = (a_{ij})_{ij}$

$$a_{k(k+1)} = \int_\Omega \nabla\psi_k \cdot \nabla\psi_{k+1} = -1/\varepsilon$$

What can we do on complex geometries ?



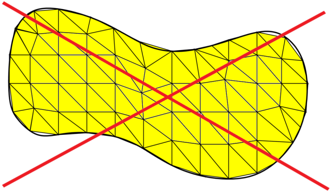
In particular how to use **hexahedral meshes**

➢ Quasi-compressible elastic models : locking effect

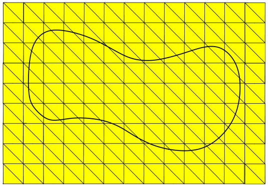
What is the looking effect ?

- Theoretical continuous quasi-compressible models "converge" to theoretical continuous incompressible models
- False for numerical models with tetrahedrons !
  - ➢ not enough degree of freedom

Alternative : Fictitious domain methods

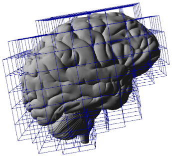


⇒



Advantages

- No need to mesh
- Regular cells



Difficulties

- Adapt the **weak formulation**
- **Conditioning** of the FE matrix

Outline

- The "standard" Finite Element Method
- Geometrical constraint on the mesh
- **Previous Fictitious domain Methods**
- ϕ-FEM
- Conclusions and perspectives

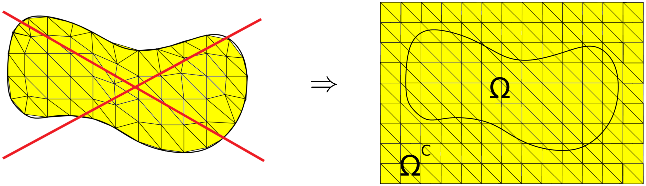
Previous works, first works

References

Saul'ev 63' (Dirichlet), Astrakhantsev 78' (Neumann),
Glowinski 92'(first proof)

Formal idea

Total energy = (energy on Ω) + $\varepsilon\times$(energy on Ω^c) $\qquad (\varepsilon << 1)$



Advantage

Non-conform mesh (complex and time varying geometry)

Difficulty

Discontinuity, **large FE matrix** and **bad cond. number**

References

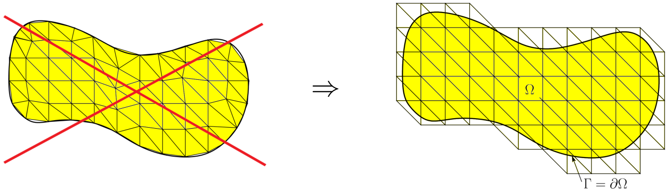
Moes-Bechet-Tourbier 2006, Haslinger-Renard 2009

Formal idea

Cut shape function : $\psi_k \longrightarrow \psi_k \mathbb{1}_\Omega$

Boundary condition on Γ : Lagrange multiplier

Conditioning of the matrix : stabilization on the boundary



Advantage

Small FE matrix

Good conditioning number

Difficulty

Non-classical shape functions and **discontinuity** in the integrals

References

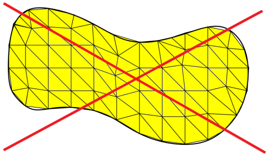
Burman et al 2010-2014

Formal idea

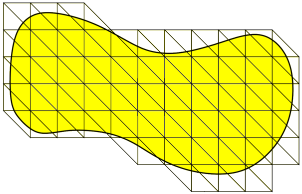
Partial integration on the cell near the boundary

Lagrange multiplier or penalization for the boundary conditions

Conditioning of the matrix : stabilization on the boundary



$\Rightarrow$



Advantage

Standard shape functions

Difficulty

- **Integral on the real boundary, cut integral**.
- $\mathbb{P}_k$ formulation

Formal idea (Dirichlet)

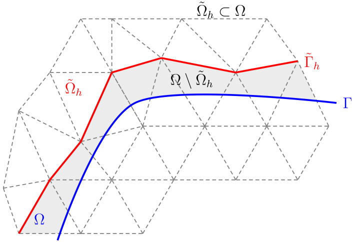
Taylor expansion of the bound. cond.

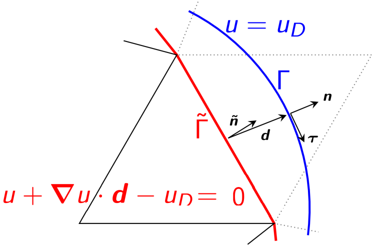
Real boundary condition on Γ : $\boldsymbol{u = u_D}$

Discrete bound. cond. on $\widetilde{\Gamma}$: $\boldsymbol{u + \nabla u \cdot d = u_D}$

Reference

Main-Scovazzi 2017





Advantage

No cut integral

Difficulty

- Construction of d
- $\mathbb{P}_k$, **Neumann conditions**

Outline

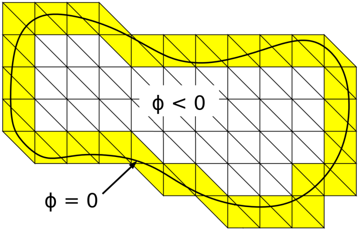
- The "standard" Finite Element Method
- Geometrical constraint on the mesh
- Previous Fictitious domain Methods
- **ϕ-FEM**
- Conclusions and perspectives

What is the idea of ϕ-FEM ?

Hypothesis :

Assume that Ω and Γ are given by a **level-set** function ϕ :

$$\Omega := \{\phi < 0\} \text{ and } \Gamma := \{\phi = 0\}.$$

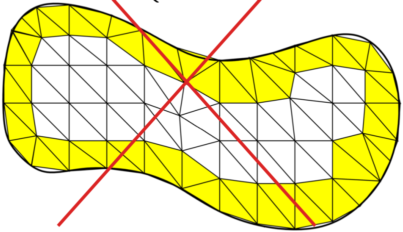


Idea of ϕ-FEM :

Include the Level-set function in the formulation
to take into account the boundary conditions

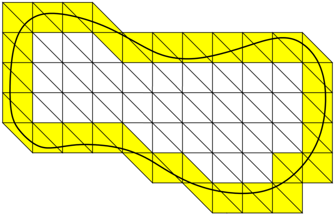
φ FEM for the Poisson Dirichlet Problem

$\exists\ ?\ u$ t.q. $\begin{cases} -\Delta u = f \\ \boldsymbol{u = 0} \textbf{ au bord} \end{cases}$



$\exists\ ?\ v$ t.q. $-\Delta(\varphi v) = f$

où $\boldsymbol{\Omega = \{\varphi < 0\}}$, $\boldsymbol{u = \varphi v}$



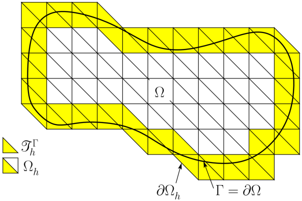
Hypothesis : Ω and Γ are given by a level-set function ϕ :

$$\Omega := \{\phi < 0\} \text{ and } \Gamma := \{\phi = 0\}.$$

ϕ-FEM formulation : Find v_h s.t.

$$\int_{\Omega_h} \nabla(\phi v_h)\cdot\nabla(\phi w_h) - \int_{\partial\Omega_h} \frac{\partial}{\partial n}(\phi v_h)\phi w_h + \text{Stab. Term} = \int_{\Omega_h} f\phi w_h \ \forall w_h \in W_h,$$

where $W_h = \{$cont. piecewise pol. functions on a the mesh$\}$.
Then $u_h := v_h\phi$ as approximation of the initial problem.

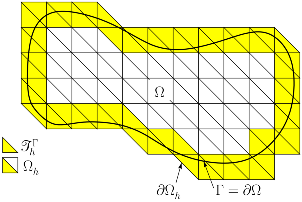


Stabilization terms (to ensure a good conditioning number)

$$\sigma h \sum_{E \in \mathcal{F}_\Gamma} \int_E \left[\frac{\partial}{\partial n}(\phi_h v_h)\right]\left[\frac{\partial}{\partial n}(\phi_h w_h)\right] + \sigma h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T (\Delta(\phi_h v_h) + f)\Delta(\phi_h w_h)$$

where $[\cdot]$ is the jump on the interface E

$$\begin{cases} \mathcal{T}_h^\Gamma = \{T \in \mathcal{T}_h : T \cap \Gamma_h \neq \varnothing\} \quad (\Gamma_h = \{\phi_h = 0\}); \\ \mathcal{F}_\Gamma = \{E \text{ (an internal edge of } \mathcal{T}_h) \text{ such that } \exists T \in \mathcal{T}_h : \\ \qquad\qquad T \cap \Gamma_h \neq \varnothing \text{ and } E \in \partial T\}. \end{cases}$$



Theorem (D.-Lozinski 2020, optimal error)

Let u be the continuous solution and v_h the ϕ-FEM solution

Deformation error *:* $|u - u_h|_{1,\Omega} \leq Ch^k \|f\|_{k,\Omega_h}$

Displacement error *:* $\|u - u_h\|_{0,\Omega} \leq Ch^{k+1/2} \|f\|_{k,\Omega_h}$

with $C = C(\phi)$ and k the polynomial order.

Proposition (Optimal conditioning)

The finite element matrix of ϕ-FEM satisfies

$$\mathit{Cond}(\mathbf{A}) \leq Ch^{-2}.$$

Domain Ω : circle of radius $\sqrt{2}/4$ centered at $(0.5, 0.5)$

Surrounding domain : $\mathcal{O} = (0, 1)^2$

Level-set function : $\phi(x, y) = (x - 1/2)^2 - (y - 1/2)^2 - 1/8$

Exact solution : $u(x, y) = \exp(x) \times \sin(2\pi y)$

Artificial external force : $f := -\Delta u$

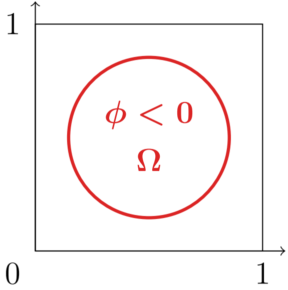
Boundary : $u_D := u(1 + \phi)$

Remark (non-homogeneous case)

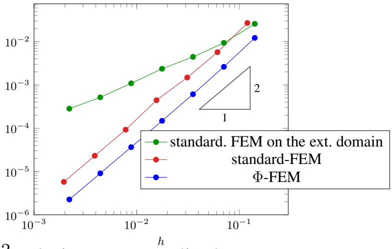
If $u := u_D$ on Γ,

we replace $\phi_h v_h$ in the scheme

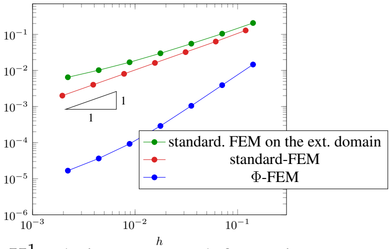
by $\phi_h v_h + u_D$ on Ω_h^Γ.



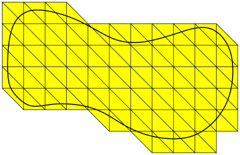
ϕ-FEM : Simulations in $\mathbb{P}_1$



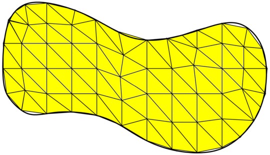
L^2 relative error $\approx$ displacement error



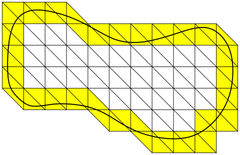
H^1 relative error $\approx$ deformation error



standard FEM on the extended domain with a perturbation (Dirichlet : $(1+\phi)u$)



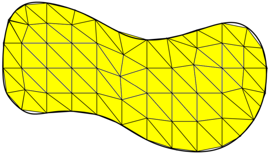
standard FEM



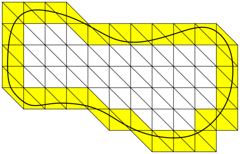
ϕ-FEM

Explanation of the numerical results

- Why better than standard FEM ?
  - ➢ Standard FEM : polygonal approximation of the boundary
  - ➢ ϕ-FEM : better approx. of the bound. with a levelset
- Numerical cost of ϕ-FEM :
  - ➢ Good point : small size of the FEM matrix
  - ➢ Bad point : quadrature more expensive
    $\rightarrow$ integral of polynomial product



standard FEM

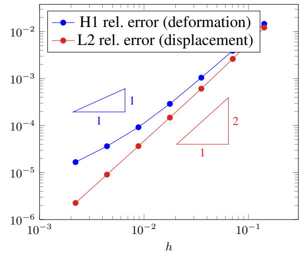


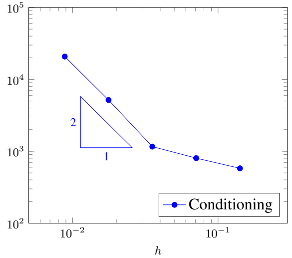
ϕ-FEM

ϕ-FEM : Stabilization and conditioning

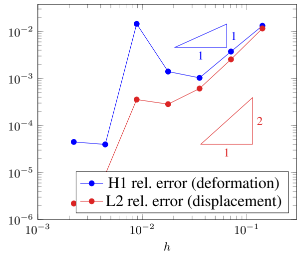
ϕ-FEM in $\mathbb{P}_1$ with stabilization	ϕ-FEM in $\mathbb{P}_1$ without stabilization

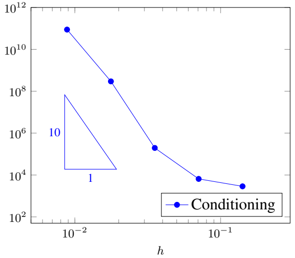
ϕ-FEM in $\mathbb{P}_1$ with stabilization



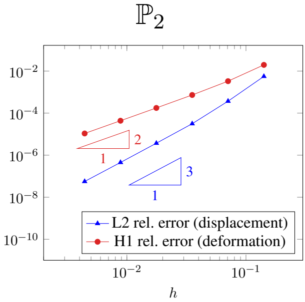


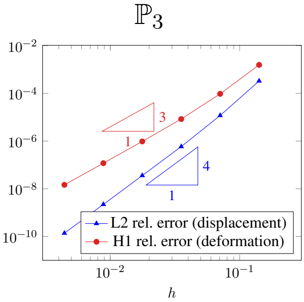
ϕ-FEM in $\mathbb{P}_1$ without stabilization





Remarque : ϕ-FEM works high polynomial orders





Neumann boundary condition

Consider the **Poisson Neumann** problem :

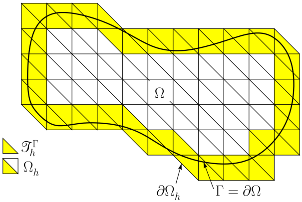
$$\begin{cases} -\Delta u + u = f, & \text{in } \Omega \\ \nabla u \cdot n = g, & \text{on } \partial\Omega \end{cases}$$

If

$$\boxed{\phi = 0 \quad \text{and} \quad \nabla\phi = n \text{ on } \Gamma,}$$

the last system is (formally) equivalent to

$$\begin{cases} y = -\nabla u, & \text{in } \mathcal{T}_h^\Gamma \\ \boldsymbol{y \cdot \nabla\phi = p\phi + g}, & \text{in } \mathcal{T}_h^\Gamma \end{cases} \quad \approx> \quad \boldsymbol{\nabla u \cdot n = g} \ \text{ on } \Gamma$$



Neumann boundary condition : weak formualtion

Find $(u_h, y_h, p_h) \in W_h^{(k)}$ such that for all $(v_h, z_h, q_h) \in W_h^{(k)}$

$$\int_{\Omega_h} \nabla u_h \cdot \nabla v_h + \int_{\Omega_h} u_h v_h + \int_{\partial\Omega_h} y_h \cdot n v_h$$

$$+\gamma_u \int_{\Omega_h^\Gamma} (y_h + \nabla u_h) \cdot (z_h + \nabla v_h)$$

$$+\frac{\gamma_p}{h^2} \int_{\Omega_h^\Gamma} (y \cdot \nabla \phi_h + \frac{1}{h} p_h \phi_h)(z \cdot \nabla \phi_h + \frac{1}{h} q_h \phi_h)$$

$$+\sigma h \int_{\Gamma_h^i} [\partial_n u] [\partial_n v_h] + \gamma_{div} \int_{\Omega_h^\Gamma} (\operatorname{div} y_h + u_h)(\operatorname{div} z_h + v_h)$$

$$= \int_{\Omega_h} f v_h + \frac{\gamma_p}{h^2} \int_{\Omega_h^\Gamma} g(z \cdot \nabla \phi_h + \frac{1}{h} q_h \phi_h) + \gamma_{div} \int_{\Omega_h^\Gamma} f(\operatorname{div} z_h + v_h),$$

where φ_h is the Lagrange interpolation of ϕ of order l and

$$W_h = \{(u_h, y_h, p_h) \in C^0(\Omega_h) \times C^0(\Omega_h^\Gamma)^d \times L^2(\Omega_h^\Gamma) :$$
$$(u_h, y_h, p_h)_K \in \mathbb{P}_k \times (\mathbb{P}_k)^d \times \mathbb{P}_{k-1}\}$$

Neumann : optimal convergence

Theorem (D.-Lleras-Lozinski 2021, optimal error)

Let u be the continuous solution and u_h the ϕ-FEM solution

Deformation error : $|u - u_h|_{1,\Omega} \leq Ch^k(\|f\|_{k,\Omega_h} + \|g\|_{k+1,\Omega_h^\Gamma})$

Displacement error : $\|u - u_h\|_{0,\Omega} \leq Ch^{k+1/2}(\|f\|_{k,\Omega_h} + \|g\|_{k+1,\Omega_h^\Gamma})$

with $C = C(\phi)$ and k the polynomial order.

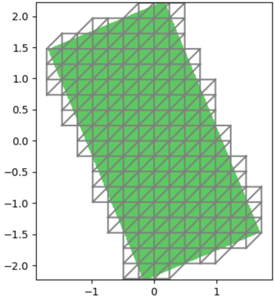
Proposition (Optimal conditioning)

The finite element matrix of ϕ-FEM satisfies

$$Cond(\mathbf{A}) \leq Ch^{-2}.$$

ϕ-FEM Neumann Poisson

- **Level set** function : distance to the boundary
- **Exact solution** : $u(x,y) := \sin(x)\exp(y)$
- **Source** term : $f := -\Delta u + u$
- Extrapolated **Neumann** boundary condition : $\tilde{g} = \frac{\nabla u \cdot \nabla \phi}{|\nabla \phi|} + u\phi$



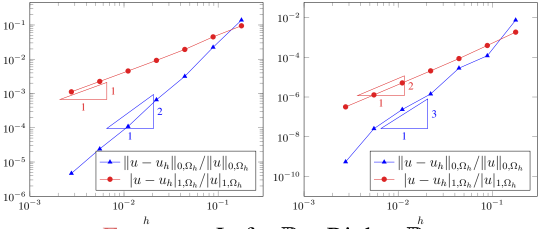


FIGURE – Left : $\mathbb{P}_1$; Right : $\mathbb{P}_2$

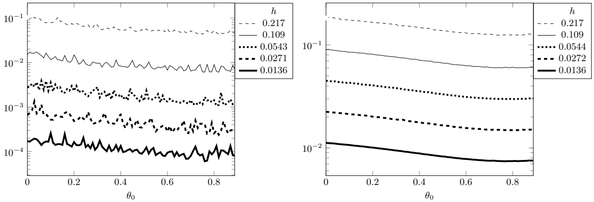


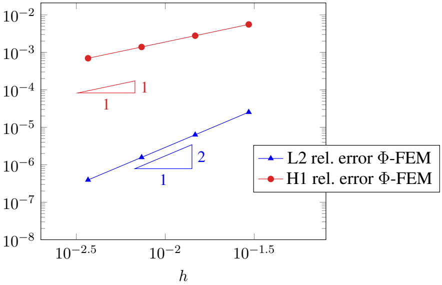
FIGURE – Sensitivity to the rotation in ϕ-FEM, $k = 1$ and $l = 3$. Left : L^2 relative error ; Right : H^1 relative error.

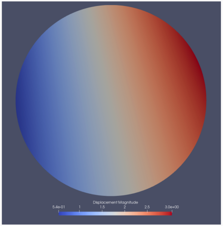
Model

$$\begin{cases} -\text{div}(\sigma(u)) = f, & \text{in } \Omega, \\ u = u_D, & \text{on } \Gamma \end{cases}$$

ϕ-FEM model

$$-\text{div}(\sigma(\phi v + u_D)) = f, \text{ in } \Omega,$$





[Duprez-Roussel]

Results

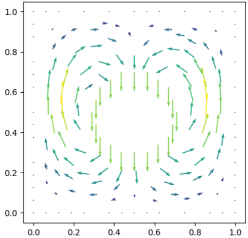
For the Poisson Dirichlet/Neumann problem :

- Optimal convergence
- Discrete problem well conditioned
- Simple implementation : standard shape functions
- Formulation available for any order of approximation

ϕ-FEM works for the elasticity equations

Work in progress

- Dynamic system
- Fluid structure
- Implementation of elastic models in Sofa



Thanks for your attention !